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Quick Computation of $[C], [L], [G]$, and $[R]$ Matrices of Multiconductor and Multilayered Transmission Systems

Gonzalo Plaza, Francisco Mesa, and Manuel Horno

***Abstract*—This paper presents a general scheme to compute the four characteristic matrices, $[C], [G], [L]$ and $[R]$, of a multilayered and multiconductor transmission line with arbitrary cross section conductors under quasi-TEM approach and strong skin effect regime. The conductors are modeled as a set of infinitesimally thin strips following the M-strip model. An spectral domain approach (SDA) has been employed, paying special attention to the efficient computation of the spectral tails. Conductor losses are considered via the incremental inductance rule extended to the multiconductor case.**

I. Introduction

The computation of the TEM (or quasi-TEM) parameters of multiconductor transmission lines having arbitrary cross section conductors embedded in a layered medium is basic for the design and analysis of a variety of technological problems ranging from microwave integrated/printed circuits to high speed interconnects. Several methods have been gradually reported in the literature to compute these parameters, although general treatments have been only provided recently. Some of these general methods, following different techniques, have been reported in [1]–[3] and [4]. In [5] and [6] the authors proposed and developed the M-strips model to analyze arbitrary cross section perfect conductors in a multilayered medium without restriction on those dielectric and magnetic properties compatible with the quasi-TEM approach.

In the present work, we present a new mixed spectral/spatial domain approach to compute the characteristic matrices of transmission lines with arbitrarily cross section conductors (in laterally open environment) using the M-strips model. We have attained a good numerical efficiency by combining the complex images technique [7] with a nontrivial extension of the guide lines suggested in [8] to evaluate the required inner products and convolutions and the use of recurrence relationships [6]. We have also incorporated the study of conductor losses under strong skin effect regime by computing the resistance matrix via an extension of the incremental inductance rule of Wheeler to a multiconductor line [9].

II. Analysis

Following the M-strip model [5], each original conductor with arbitrary cross section is modeled as a set of infinitesimally thin strips at the same potential and circumscribed to the contour of the original conductor. The corresponding spectral integral equation is solved via the Galerkin's method (using as spatial basis function the Chebyshev's polynomials weighed by the Maxwell's distribution). The spectral integrals appearing in the Galerkin's matrix show the following form

$$\Gamma_{ipn}^{jqm} = \int_{-\infty}^{+\infty} J_n\left(\frac{w_p k_x}{2}\right) G_{ij}(k_x) J_m\left(\frac{w_q k_x}{2}\right) e^{jk_x s}\, dk_x \tag{1}$$

where the subscripts p and q are used to number the strips in the model, $w_{p(q)}$ is the width of each strip; $s = c_q - c_p$ with $c_{p(q)}$ being the abscisa of their centers; J_n is the Bessel function of order n and $G_{ij}(k_x)$ is the spectral Green's function for a source line at the jth metallizated level (where qth strip is placed) and field points at the ith level (where pth strip is placed).

Since the numerical efficiency of the SDA is strongly determined by the fast and accurate computation of the spectral integrals, we have paid an especial attention to accelerate its computation because we face to a very slow convergence caused by extremely close strips. We have used the asymptotic integration scheme shown in [6], where the asymptotic behavior of the spectral Green's function, G_{ij}^{∞}, is now expressed following the complex images technique [7]

$$G_{ij}^{\infty}(k_x) = \sum_{n=1}^{N} A_n^{ij} \frac{\exp(\Omega_n^{ij} k_x)}{k_x}. \tag{2}$$

Expansion (2) provides a very good fitting of the spectral Green's function and thus the integrals involving $G - G^{\infty}$ converge quickly. In consequence, the efficiency of the proposed integration technique lies basically in the computation of the integral tails. The generic form, except for constants, of these integral tails is

$$I_n^m = \int_u^{\infty} J_n(ak_x) J_m(bk_x) \frac{e^{(-\beta + jd)k_x}}{k_x}\, dk_x \tag{3}$$

where u is a suitable value to start using the asymptotic behavior; a and b are the semiwidths of the strips; $\beta = -\mathrm{Re}(\Omega)$ and $d = s + \mathrm{Im}(\Omega)$ (Ω stands for any of the complex exponents in (2)). The new distance, d, can be seen as a modified distance between the centers of the strips in the integrals tails.

The integral tails (3) are computed by reversing them to the spatial domain via Parseval's theorem. The use of Parseval's theorem requires a previous extension of the tails from $-\infty$ to $+\infty$. This extension is readily made by multiplying the integrand in (3) by the step function $H(k_x - u)$ and so, the following spatial domain integral is obtained

$$I_n^m = \frac{j^n(-j)^m}{\pi^2} \int_{-1}^{1} \frac{T_n(\gamma)}{\sqrt{1-\gamma^2}} \cdot \left[\int_{-1}^{1} \frac{T_m(\alpha)}{\sqrt{1-\alpha^2}} E_1(u\zeta)\, d\alpha\right] d\gamma \tag{4}$$

where $\zeta = \beta + j(a\gamma - b\alpha - d)$. α and γ appear after changing the integration variable (both in the convolution integral in α- and the inner product integral in γ-) into the interval $(-1, 1)$ and $E_1(u\zeta)$ stands for the exponential integral function, which can be expanded as shown in [10]. The convolution product in expression (4) might be regarded as a complex *potential* originated by a strip of width $2b$ (*source-strip*) over a second strip of width $2a$ (*observer-strip*) placed at a height β above the first one, and with its center laterally separated a distance d. We will express ourselves in these terms to simplify the descriptions in the below analysis.

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Now, we consider separately the performance of the two integrals appearing in (4). First, the convolution product (denoted by $C_m(\gamma)$ from now on) is handled to extract analytically the possible singularities. In accordance to the serie expansion of the exponential integral, this convolution can be expressed as

$$C_m(\gamma) = -(\gamma_e + \ln\ u)\pi\delta_{m0} - R_m(\gamma) - S_m(\gamma) \tag{5}$$

with

$$R_m(\gamma) = \int_{-1}^{1} \frac{T_m(\alpha)}{\sqrt{1-\alpha^2}} \sum_{k=1}^{\infty} \frac{(-u\zeta)^k}{kk!}\, d\alpha \tag{6}$$

$$S_m(\gamma) = \int_{-1}^{1} \frac{T_m(\alpha)}{\sqrt{1-\alpha^2}} \ln(\zeta)\, d\alpha \tag{7}$$

where γ_e is the Euler's constant and δ_{mn} is the delta of Kronecker. The integrand of $R_m(\gamma)$ shows the proper form to be evaluated by means of a Chebyshev's quadrature, and 5-6 quadrature points suffice if we choose $u \sim 0.5/\max(a,b)$. Greater values of u are not advisable because it would imply a greater number of quadrature points.

The integrand of $S_m(\gamma)$ has a singularity when $\beta = d = 0$ and $a = b$ (we refer to this case as the *singular case*). This situation occurs when the integral S_m involves the effect of a strip over itself. Nevertheless, the use of the M-strips model makes the integrand of S_m be *quasi–singular* even for different-strips cases since the strips can overlap ($|d| < a+b$) and be very close ($\beta < a, b$). Both cases, the singular and *quasi-singular* ones, usually require a numerical quadrature of an impracticable high order to give accuracy and therefore it would be very desirable to find analytical expressions for $S_m(\gamma)$.

In the general nonsingular case (i.e., including the quasi-singular case), analytical expressions for $S_m(\gamma)$ can be obtained using complex variable techniques

$$S_m(\gamma) = \left[-j\frac{\pi^2}{2} + \pi \ln \frac{b(\xi-1)}{(1+\eta)^2}\right]\delta_{m0} - \pi\frac{\eta^m}{m}(1-\delta_{m0}) \tag{8}$$

with $\xi = (d - a\gamma + j\beta)/b$ and $\eta = -\xi \pm \sqrt{\xi^2 - 1}$, where the sign in the squared root must be chosen in such a way that $|\eta| < 1$. Once the quasi-singularities have been treated, (3) can be rewritten as

$$I_n^m = j^n(-j)^m \Bigg\{ -\Big(\gamma_e + \ln\ u - j\frac{\pi}{2}\Big)\delta_{m0}\delta_{n0} - \frac{1}{\pi^2}\int_{-1}^{1} \frac{T_n(\gamma)}{\sqrt{1-\gamma^2}} R_m(\gamma)\, d\gamma + B_n^m \Bigg\} \tag{9}$$

with

$$B_n^0 = -\ln \frac{d-b+a+j\beta}{(1+\rho)^2}\delta_{n0} + \frac{\rho^n}{n}(1-\delta_{n0}) + \frac{2}{\pi}\int_{-1}^{1} \frac{T_n(\gamma)}{\sqrt{1-\gamma^2}} \ln(1+\eta)\, d\gamma \tag{10}$$

$$B_n^m = -\frac{1}{\pi m}\int_{-1}^{1} \frac{T_n(\gamma)}{\sqrt{1-\gamma^2}} \eta^m\, d\gamma \quad (m \neq 0)$$

$$\rho = \left(\frac{d-b+j\beta}{a}\right) \pm \sqrt{\left(\frac{d-b+j\beta}{a}\right)^2 - 1} \tag{11}$$

where the sign in the square root must be chosen in such a way that $|\rho| < 1$. Analytical expressions for the singular case ($\beta = d = 0$ and $a = b$) (except for integrals involving R_m) can be obtained as the limit of (9)–(11) when β tends to zero. It should be noted that spectral tails involving order higher than 1 can be evaluated starting from I_0^0, I_0^1, I_1^0 and I_1^1 [6].

TABLE I

MATRICES $[C], [L]$ AND NORMALIZED RESISTANCE $[R]/R_s$ FOR A TRANSMISSION LINE WITH TWO RECTANGULAR CONDUCTORS AND A CIRCULAR CONDUCTOR. DIMENSIONS OF THE LINE ARE IN mm. R_s: SURFACE RESISTANCE OF THE CONDUCTORS

	Ref.[1]	Ref.[2]	This work
C_{11}	124.4	125.86	125.04
C_{12}	-13.00	-13.125	-12.960
C_{13}	-68.25	-69.555	-69.27
C_{22}	33.40	34.101	33.875
C_{23}	-7.196	-7.1818	-7.2044
C_{33}	352.3	357.62	357.15

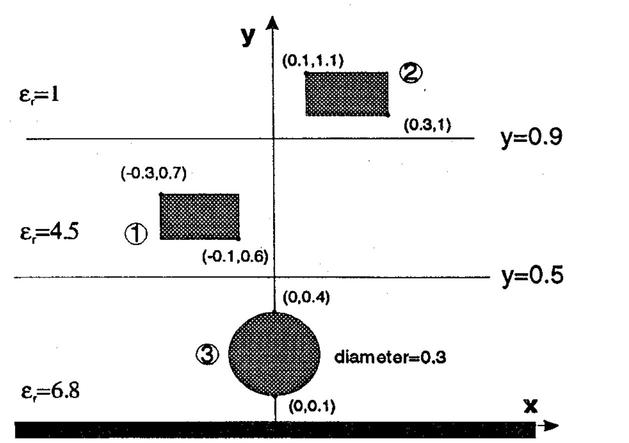
	This work
r_{11}	2454.9
r_{12}	110.1
r_{13}	-161.0
r_{22}	2338.6
r_{23}	-150.2
r_{33}	547.2

	Ref.[1]	Ref.[2]	This work
L_{11}	496.5	491.90	494.81
L_{12}	199.6	198.88	199.17
L_{13}	118.3	117.50	117.84
L_{22}	616.3	612.83	615.55
L_{23}	77.28	76.781	77.04
L_{33}	233.1	229.94	230.21

C_{ij} (pF/m)

L_{ij} (nH/m)

$r_{ij} = R_{ij}/R_s$ (1/m)



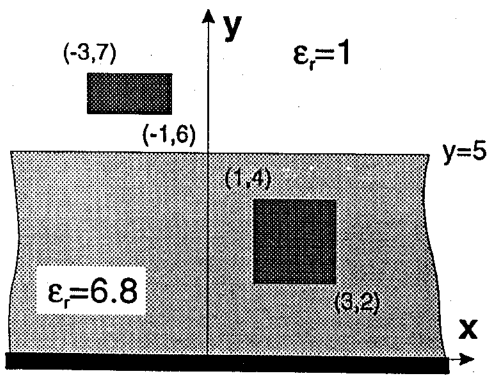
In expressions (9)–(11) for the nonsingular case, two different types of integrals remain: 1) integrals involving $R_m(\gamma)$ and 2) integrals involving $\sqrt{(1-\xi^2)}$ (note that function η includes this function). The first type of integrals can be accurately computed via a Chebyshev's quadrature, using the same criterion as previously mentioned to compute $R_m(\gamma)$. In the second type of integrals, the function's slope of $\sqrt{(1-\xi^2)}$ presents abrupt changes at $\gamma_{1,2} = (d \pm b)/a$ when the strips are very close $\beta/b < 1$. This means a large number of quadrature points if any of these points of abrupt change lies in the integration interval $\gamma \in (-1, 1)$. When both points of abrupt change belong to $(-1, 1)$, we can readily take them out of the interval by interchanging the role of the strips, that is, choosing b as the semiwidth of the largest strip. For example, if $\beta = 0.001, a = 0.5, b = 0.25$ and $d = 0$, approximately a 1600 points Chebyshev's quadrature would be necessary to get 5 significant digits. Choosing now $a = 0.25, b = 0.5$, 8 significant digits are obtained

TABLE II

VALUES OF THE CAPACITANCE COEFFICIENTS OF A TRANSMISSION LINE CONTAINING TWO RECTANGULAR CONDUCTORS AS A FUNCTION OF THE NUMBER OF STRIPS USED TO MODEL THE CONDUCTORS (N_s) AND THE NUMBER OF BASIS FUNCTIONS EMPLOYED TO EXPAND THE CHARGE DENSITY ON EACH STRIP

	Number of Basis Functions											
	5				7				9			
N_s	C_{22}	C_{12}	C_{11}	CPU	C_{22}	C_{12}	C_{11}	CPU	C_{22}	C_{12}	C_{11}	CPU
2	202.8	-14.18	36.37	0.3	202.8	-14.18	36.37	0.4	-	-	-	0.5
4	210.5	-15.15	37.24	0.8	210.5	-15.15	37.24	1.1	-	-	-	1.3
6	212.5	-15.39	37.48	2.0	212.5	-15.39	37.48	2.5	-	-	-	2.9
8	213.5	-15.51	37.59	4.0	213.5	-15.51	37.60	5.6	-	-	-	7.0
10	214.1	-15.57	37.65	7.0	214.1	-15.58	37.67	9.6	214.1	-15.59	37.67	13
12	214.4	-15.61	37.68	12	214.5	-15.63	37.71	18	214.5	-15.63	37.72	22
14	214.7	-15.64	37.71	18	214.8	-15.66	37.74	25	214.8	-15.66	37.75	34
16	214.9	-15.66	37.73	28	215.0	-15.68	37.77	36	215.0	-15.69	37.78	50
18	215.0	-15.67	37.74	40	215.1	-15.70	37.78	52	215.1	-15.71	37.79	68
								Ref.[1]	209.9	-15.62	36.51	
								Ref.[4]	206.4	-15.50	35.21	

C_{ij} (pF/m) CPU (sec)



by means of an 8 points Chebyshev's quadrature. When only one of these points of abrupt change belongs to the integration interval, the above solution does not apply (the point of abrupt change would remain in the interval). Nevertheless, we can significantly reduce the total number of quadrature points by dividing the integration interval into two new integration intervals separated by the point of abrupt change and then using a Gauss-Legendre's quadrature in each new interval. For instance, a Chebyshev's quadrature, when $a = b = 0.5, d = 0.5, \beta/a = 0.002$, would require 2247 points to obtain similar accuracy as that achieved with two Gauss-Legendre's quadrature of 80 points after dividing the interval.

III. NUMERICAL RESULTS

In this section, we present several numerical results obtained via a computer code implemented according to the scheme proposed above. First and for comparison, Table I shows our results in good agreement with some results previously reported in [1] and [2] for the capacitance and inductance elements of a three-conductors transmission line containing a conductor of circular cross section. We also report novel results about the normalized-resistance elements of this structure. Our results in Table I were obtained modelling the circular conductor by 15 thin strips, the rectangular conductors by 11, and using five basis functions for each thin strip.

After this first comparison, Table II shows a numerical study about the convergence of our model with respect to the number of basis functions and the number of thin strips, N_s, in each rectangular conductors. An estimation of the CPU time consumed by a HP 9000/730 work-station is also provided in Table II. It can be seen how the different capacitance values converge properly as the number of basis functions and thin strips increases, although the convergence with respect to the the number of basis functions is more stable than with respect to the number of thin strips. Nevertheless in the present case, relative errors less than 1% are achieved just with eight thin strips for each rectangular conductor. In many technological cases (i.e., those concerning *thick* strips rather than rectangular bars or circular wires), no more than three/four thin strips would be usually required.

As a third example, we have computed the resistance of a rectangular conductor ($2a \times a$) over a perfect conductor ground plane at a distance a as that analyzed in [3, Table III]. In this case, our result ($r = aR/R_s = 0.23673$) agrees very well with that reported in [3] ($r = 0.23753$). We have modeled the rectangular conductor by 15 thin strip and the charge density was expanded into 5 basis functions. The resistance is then computed applying the incremental inductance rule. The good agreement between the results highlights the suitability of the M-strips model to compute conductor losses under strong skin effect.

IV. Conclusion

We have presented a semi-analytical analysis of general multilayered multiconductor transmission lines with arbitrary cross section conductors using the M-strips model. This procedure has proved its ability to compute in a fast and accurate way the characteristic matrices of the analyzed transmission lines. It has been also shown how the M-strip model combined with the Wheeler's incremental inductance rule yields sufficiently accurate results for the conductor losses assuming strong skin effect. The studied examples have shown that rectangular conductors and even circular conductors can be efficiently modeled with a reasonable number of thin strips. This latter fact and the enhanced numerical treatment here applied suggests that our scheme may be used as a good basis for CAD of general transmission lines.

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A Numerical Method of Evaluating Electromagnetic Fields in a Generalized Anisotropic Medium

Hung-Yu Yang

***Abstract*—A transition matrix method is commonly used to deal with the problems of either plane-wave scattering from or the Green's function of a generalized anisotropic medium. This method, although rigorous analytically, introduces numerical breakdown, when the layers are electrically thick and the waves are evanescent. A variable transformation method is developed to deal with the exponentially-growing terms associated with exponential-matrix method. The proposed scheme is suitable for the numerical analysis of generalized anisotropic layers including ferrites, magneto-plasmas, chiral layers, and bianistropic layers.**

I. Introduction

In the past, there have been considerable interest in the investigation of the interaction of electromagnetic waves with anisotropic materials. The classical formulation for antennas on layered media employing a combination of TE and TM vector potential functions limits the applications to isotropic or uniaxial media. In recent years, the interest in the technology of printed circuit elements on anisotropic substrates has stirred the investigation of electromagnetic waves interaction with generalized anisotropic layered media. A spectral exponential 4 × 4 matrix method has been developed to deal with embedded dipoles in or scattering from a layered generalized anisotropic structure [1]–[6]. The exponential matrix method is a useful numerical method in dealing with waves in media with arbitrary anisotropy. There, the derivation of analytic form of waves is often complicate and tedious if not impossible. Most published research in the area of electromagnetic waves in layered anisotropic media dealt with the analytic aspect of the problem. The full-wave numerical implementation of the spectral matrix method has been applied for microstrip transmission-lines [7]–[9] and for printed antennas [10], [11]. A critical step in the exponential-matrix method is to develop the transition matrices which relate the electromagnetic fields at one planar interface to the others. This method although elegant analytically has inherent deficiency in the numerical implementation. Problems arise when the wave numbers in the direction of inhomogeneity are complex-valued. If the layers are electrically thick enough, the transition matrices become numerically singular and can no longer pass the complete information of fields. The physical explanation is that from one layer interface to another, part of the waves die out before reaching the interface. The remaining propagating waves are degenerate.

As a result, the 4 × 4 transition matrix is singular. This problem is particularly serious in dealing with antennas and circuits on anisotropic media, where the plane wave representations of fields always include the evanescent plane wave spectrum. This numerical singularity (or overflow) problem occurs often in dealing with isotropic or uniaxial media, where the problem is overcome in the analytic formulation, by normalizing the variables such that we deal with the "tanh" functions instead of the "cosh" or "sinh" functions.

In this paper, a scheme utilizing variable transformation is developed. The idea is to extract the large exponential terms in

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